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Differential invariants on the bundles of linear frames

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Abstract. *A study is made of the rings A_r of r-order differential invariants of linear frames on a differentiable manifold X with respect to the Lie algebra of the vector fields of X. It is demostrated that, locally, such rings are differentiably finitely generated and canonical bases are determined. The global structure of the rings A_r and that of the subrings $A'_r \subseteq A_r$ of differential invariants under the group of the diffeomorphisms of X are determined. As an application of the theory the problem of local equivalence of complete parallelisms is solved, demonstrating that the equality of the basic differential invariants of two fields of linear frames are sufficient conditions for their formal equivalence (and hence, analytical).*

INTRODUCTION

Study of the differential invariants of «geometric objects» on a differentiable manifold with respect to the group of the diffeomorphisms of the manifold is a classic problem in differential geometry. As is known, the equality of the invariants associated with two of such objects are the natural necessary conditions for their local equivalence; this problem is recognized as being fundamental to this discipline. Given a closed subgroup G of $GL(n, \mathbb{R})$, if $p_G : E \to X$ is the fibre bundle whose sections are the G-structures on a differentiable manifold X of dimension n ([6]) and $J^r(E)$ is the corresponding r-jet bundle, an r-order

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differential invariant of the G-structures on X is defined as a function $f \in C^\infty(J^r E)$ such that for all diffeomorphisms τ of X one has $f \circ \bar{\tau}^{(r)} = f$, where $\bar{\tau}^{(r)}$ is the r-jet prolongation of the natural lifting $\bar{\tau}$ of τ to E. The two basic questions of the theory are now: to study under which conditions the rings defined by such invariants are differentiably finitely generated and to attempt to fund explicit bases for them that have geometric meaning.

The infinitesimal version of the theory – that is: invariance with respect to the Lie algebra of vector fields of X – is clear and can be inserted in the general theory of differential invariants with respect to an infinitesimal Lie pseudo-group, a classic theory that in the 70's decade received considerable thrust by application of the cohomological methods of Spencer (see for example [8]).

In this version, the rings $A_r(G)$ of r-order differential invariants of G-structures are no more than the first integrals of the involutive differential systems $\mathcal{M}^r(G)$ generated in $J^r(E)$ by the r-jet prolongations $\bar{D}_{(r)}$ of the natural liftings $\bar{D}$ to E of the vector fields D of X. The crucial point of the problem is now that the differential system $\mathcal{M}^r(G)$, generally, are not regular over the whole of $J^r(E)$. Unfortunately, these methods, whose interest for the theory of invariant differential equations by an infinitesimal Lie pseudogroup is undeniable, are not appropriate for the case in question, precisely because the invariance Lie algebra of this problem is too broad.

However, there seems to have been more successful in the relation recently found between these invariants and gauge theories, a theory whose geometric formulation was also established in the decade of the seventies with apparently no reference to differential invariant theory ([1, 2, 3]). Indeed, careful examination of one of the formalisms developed to understand General Relativity as a gauge theory – the formalism of T.W. Kibble [5] – suggests that differential invariants of G-structures should be considered as invariants of linear frames according to the following idea ([4]):

If $\pi : L(X) \to E$ is the principal G-bundle over E defined by the action of G on the bundle $p : L(X) \to X$ of the linear frames of X ([5]), then their r-jet prolongation defines a submersion $J^r(\pi) : J^r(L(X)) \to J^r(E)$ that induces an injection

$$A_r(G) \xhookrightarrow{J^r(\pi)^*} A_r$$

of the ring $A_r(G)$ that one wishes to study in the ring A_r of the corresponding r-order differential invariants of linear frames over X (that is, the $\{e\}$-structures over X, e = identity of $GL(n, \mathbb{R})$). The image subring $J^r(\pi)^* A_r(G)$ is identified with the first integrals of the differential system $\mathcal{M}^r + V^r(G)$, where $\mathcal{M}^r$ is the differential system defined by the r-jet prolongations of the natural liftings to $L(X)$ of the vector fields of X, and $V^r(G)$ is that generated by the $J^r(\pi)$-

vertical vector fields of $J^r(L(X))$. The formalism of T.W. Kibble - which corresponds to the case $X = \mathbb{R}^4$, $G = 0(3, 1)$ and where $\mathcal{M}^r + V^r(G)$ is simply the r^{th} derivative of the gauge algebra of the natural representation over the linear frames («vierbeins») of the Lie algebra of the inhomogeneous Lorentz group – suggests that the differential system $\mathcal{M}^r + V^r(G)$ may be easier to deal with than the original system $\mathcal{M}^r(G)$. It is exactly on this substitution that we base our treatment of the differential invariants of G-structures and in this first article we begin with the simplest case possible, $G = \{e\}$, which we logically believe should be used to start such a program.

The work is divided into three well separated parts. In the first two (§ § 1, 2, 3 and 4) we determine the rings A_r of differential invariants of frames with respect to the Lie algebra of the vector fields of X. In the third part (§ 5), the same is done for the rings $A'_r \subseteq A_r$ of invariants under the group of diffeomorphisms of X.

The solution to the problem cannot be more complete and satisfactory. The involutive differential system $\mathcal{M}^r$ is regular over $J^r(L(X))$ (Theorem 2.2) which allows us to conclude, by application of the Frobenius theorem, that the rings of invariants A_r are differentiably finitely generated on a neighbourhood of every point of $J^r(L(X))$. Additionally, from the notion of torsion of a linear connection, it is possible to define canonical bases of the rings A_r that allow us to solve in a very simple geometric fashion the second main problem of the theory of differential invariants in this case (Theorem 4.8). One fundamental ingredient for the explicit integration thus carried out is the existence of n vector fields $\mathbb{D}_1, \ldots, \mathbb{D}_n$ canonically defined on $J^\infty(L(X))$, which allow us to construct, from each differential invariant f of order r, n differential invariants $\mathbb{D}_1 f, \ldots, \mathbb{D}_n f$ of order $r + 1$ (§ 3).

Finally, Theorem 5.7 provides the global structures of the rings of invariants A_r and A'_r. It is also demonstrated (Theorem 5.5) that the bases of invariants obtained not only provide the necessary conditions for the local equivalence of fields of linear frames, but also sufficient conditions for the formal equivalence. The latter, which implies as usual sufficiency in the analytical case, is not true at the differentiable level.

1. PRELIMINARIES AND NOTATIONS

(1.1). In what follows, X will stand for a n-dimensional connected C^∞ manifold and $p : L(X) \to X$ will denote its bundle of linear frames. $L(X)$ is a principal fibre bundle over X whose structure group is the full linear group $GL(n, \mathbb{R})$. This bundle is endowed with a $\mathbb{R}^n$-valued 1-form $\theta = (\theta_1, \ldots, \theta_n)$, the so-called *canonical form* of $L(X)$. Any coordinate system $x_1, \ldots, x_n$ on an open subset U of X induces a coordinate system (x_i, z_{ij}), $1 \leqslant i, j \leqslant n$, on $p^{-1}(U)$ by the formula

$$(1.1.1)\qquad u = ((\partial/\partial x_1)_x, \ldots, (\partial/\partial x_n)_x) \cdot (z_{ij}(u)), \quad u \in p^{-1}(U)$$

The canonical form is given in terms of an induced system by

$$(1.1.2)\qquad \theta_i = \sum_{j=1}^{n} z^{ij}\, dx_j, \quad 1 \leqslant i \leqslant n,$$

where $(z^{ij}) = (z_{ij})^{-1}$ stands for the inverse matrix.

Any diffeomorphism τ of X induces an automorphism $\tilde{\tau}$ of $L(X)$ mapping the frame $u = (D^1_x, \ldots, D^n_x)$ at x into the frame $\tau(u) = (\tau_* D^1_x, \ldots, \tau_* D^n_x)$ at $\tau(x)$. If D is a vector field on X, we denote by $\tau \cdot D$ the vector field defined by $(\tau \cdot D)_x = \tau_*(D_{\tau^{-1}(x)})$. More generally, if P is a differential operator on X, we denote by $\tau \cdot P$ the differential operator of the same order given by $(\tau \cdot P)(f) = P(f \circ \tau) \circ \tau^{-1}$, $f \in C^\infty(X)$. If Q is another differential operator, then $\tau \cdot (P \circ Q) = (\tau \cdot P) \circ (\tau \cdot Q)$.

PROPOSITION (1.1.3) (Cf. [6], Ch. VI, Prop. 2.1). *Given a vector field D on X, three exists a unique vector field $\tilde{D}$ on $L(X)$ such that*

(a) $\tilde{D}$ is p-projectable onto D,

(b) $L_{\tilde{D}}\theta = 0$.

Furthermore, $\tilde{D}$ satisfies the following properties:

(c) If the local 1-parameter group τ_t induces D, then $\tilde{\tau}_t$ induces $\tilde{D}$; hence, $\tilde{D}$ is invariant under right translations, i.e., $R_A \cdot \tilde{D} = \tilde{D}$ for all $A \in GL(n, \mathbb{R})$.

(d) The mapping $D \to \tilde{D}$ is a $\mathbb{R}$-linear injection of Lie algebras, thus $[D_1, D_2]^\sim = [\tilde{D}_1, \tilde{D}_2]$ for all vector fields D_1, D_2 on X. ■

The vector field $\tilde{D}$ is called the *natural lift* of D to $L(X)$. It is easily checked that if $D = \Sigma_i\, u_i(\partial/\partial x_i)$, then:

$$(1.1.4)\qquad \tilde{D} = \sum_i u_i(\partial/\partial x_i) + \sum_{i,j} u_{ij}(\partial/\partial z_{ij}),$$

$$u_{ij} = \sum_h z_{hj}(\partial u_i/\partial x_h).$$

(1.2). Let $p: Y \to X$ be an arbitrary fibred manifold. We denote by $p_{(r)}: J^r(Y) \to X$ the r-jet bundle of local sections of p, and by $j^r s$ the r-jet extension of a local section s of p. Moreover, $\pi^r_k: J^r(Y) \to J^k(Y)$, $r \geqslant k$, stands for the canonical projection. Let m be the dimension of the fibres of p, so that

$\dim Y = m + n$. Each fibred coordinate system (x_j, y_i), $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, for the projection p induces a coordinate system (x_j, y^i_α), $|\alpha| \leqslant r$, for $J^r(Y)$ defined by: $y^i_\alpha(j^r_x s) = D^\alpha(y_i \circ s)(x)$, where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Note that $y^i_0 = y_i$. We also write (j) the multi-index whose components are $(j)_i = \delta_{ij}$, $1 \leqslant i \leqslant n$. Similarly, $(jk) = (j) + (k)$, $(jk\ell) = (j) + (k) + (\ell)$, and so on. The notation $\alpha \leqslant \beta$ for multi-indices means $\alpha_i \leqslant \beta_i$ for all $i = 1, \ldots, n$.

Let $p' : Y' \to X'$ be another fibred manifold. If $f : Y \to Y'$ is a fibred mapping (that is, f maps the fibres of p into the fibres of p'), then there exists a unique differentiable mapping $\bar{f}$ making the following diagram commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & Y' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{\ \bar{f}\ } & X' \end{array}$$

When $\bar{f}$ is a diffeomorphism, we can define the induced mapping $f^{(r)} : J^r(Y) \to J^r(Y')$ by $f^{(r)}(j^r_x s) = j^r_{\bar{f}(x)}(f \circ s \circ \bar{f}^{-1})$. If $X = X'$ and $\bar{f}$ is the identity of X, we shall also write $J^r(f)$ instead of $f^{(r)}$.

We should recall that if E is a vector bundle over X, then $J^r(E)$ also has a naural structure of vector bundle over X.

(1.3). We denote by $V(Y)$ the vector sub-bundle of the p-vertical tangent vectors of $T(Y)$. We shall also write $T(Y/X)$ whenever we need to specify the base manifold.

The fibre bundle $J^r(Y)$ is endowed with a $(\pi^r_{r-1})^* \, V(J^{r-1}(Y))$-valued 1-form θ^r, the so-called *structure form* of the r-jet bundles, geometrically defined by means of the notion of vertical differentiation (see [9]), and whose local expression is:

$$\theta^r = \sum_{i=1}^{m} \sum_{|\alpha| < r} \theta^i_\alpha \otimes (\partial/\partial y^i_\alpha), \tag{1.3.1}$$

$$\theta^i_\alpha = dy^i_\alpha - \sum_{j=1}^{n} y^i_{\alpha+(j)} \, dx_j.$$

A vector field D on $J^r(Y)$ is said to be an *infinitesimal contact transformation* if for a given derivation law ∇ on $V(J^{r-1}(Y))$ there exists an endomorphism A of $(\pi^r_{r-1})^* \, V(J^{r-1}(Y))$ such that:

$$L_D \theta^r = A \circ \theta^r, \tag{1.3.2}$$

where the Lie derivative is taken with respect to ∇([7]). The definition is easily seen to be independent of the derivation law ∇ chosen, and we have:

PROPOSITION (1.3.3) ([9]). *For every vector field D on Y (not necessarily p-projectable) there exists a unique infinitesimal contact transformation $D_{(r)}$ on $J^r(Y)$ projectable onto D. Furthermore,*

(a) If D is p-projectable and τ_t is its local 1-parameter group, then $\tau_t^{(r)}$ induces $D_{(r)}$.

(b) $D_{(r)}$ is (π^r_{r-1})-projectable onto $D_{(r-1)}$ for all $r > 0$ and every vector field D on Y.

(c) The mapping $D \to D_{(r)}$ is a $\mathbb{R}$-linear injection of Lie algebras, thus $[D', D'']_{(r)} = [D'_{(r)}, D''_{(r)}]$ for all vector fields D', D'' on Y. ■

The vector field $D_{(r)}$ is called the r-order infinitesimal contact transformation associated with D.

2. DIFFERENTIAL INVARIANTS ON L(X) AND THE ASSOCIATED DIFFERENTIAL SYSTEM

By composing the natural lift of a vector field D on X to $L(X)$ with the r-order infinitesimal contact transformation associated with $\widetilde{D}$, we obtain an injection of Lie algebras: $D \to \widetilde{D} \to \widetilde{D}_{(r)}$, by virtue of (1.1.3)-(d) and (1.3.3) -(c).

DEFINITION (2.1). *A differentiable function $f : J^r(L(X)) \to \mathbb{R}$ is said to be an r-order* differential invariant *if $\widetilde{D}_{(r)} f = 0$ for every vector field D on X.*

The set of r-order differential invariants constitutes a subring A_r of $C^\infty(J^r(L(X)))$. The purpose of the present paper is to study the structure of the rings of invariants A_r.

THEOREM (2.2). *There exists a unique homomorphism of vector bundles over $J^r(L(X))$,*

$$\phi_r : p^*_{(r)} J^{r+1}(TX) \to T(J^r(L(X)))$$

such that $\phi_r(j^r_x s, j^{r+1}_x D) = \widetilde{D}_{(r)}(j^r_x s)$ for every local section s of p and every vector field D on X. Moreomer, we have:

(i) The homomorphism induced at each fibre

$$\phi_r : J^{r+1}_x(TX) \to T_{j^r_x s}(J^r(L(X)))$$

is injective for all $j^r_x s \in J^r(L(X))$*. Therefore, the image of* ϕ_r *is a vector subbundle* $Im\ \phi_r$ *of* $T(J^r(L(X)))$.

(ii) Let us denote by $\mathcal{M}^r$ *the sheaf of* $C^\infty_{J^r(L(X))}$*-modules determined by the sections of* $Im\ \phi_r$*. Then,* $\mathcal{M}^r$ *is an involutive, locally free sheaf of modules of rank* $n\binom{n+r+1}{r+1}$*, spanned over* $C^\infty_{J^r(L(X))}$ *by all vector fields* $\widetilde{D}_{(r)}$.

(iii) A differentiable function on $J^r(L(X))$ *is an r-order differential invariant if and only if it is a first integral of* $\mathcal{M}^r$*. Thus, the ring of invariants* A_r *coincides with the ring of first integrals of an involutive differential system.*

Proof. The uniqueness of ϕ_r is obvious. In order to prove its existence, it will clearly be sufficient to prove the following:

(2.2.1). *The equality* $j^{r+1}_x(D) = j^{r+1}_x(D')$ *in* $J^{r+1}(TX)$ *implies* $\widetilde{D}_{(r)}(j^r_x s) = \widetilde{D}'_{(r)}(j^r_x s)$ *for every local section s defined on a neighbourhood of* $x \in X$.

Proof of (2.2.1). With notations as in (1.1.4) and (1.2) we have:

$$\widetilde{D}_{(r)} = \sum_i u_i(\partial/\partial x_i) + \sum_{i,j} \sum_{|\alpha|=0}^{r} u^{ij}_\alpha(\partial/\partial z^{ij}_\alpha),$$

where $u^{ij}_0 = u_{ij}$, and according to (1.3.3)-(a) the rest of the functions are given by

$$u^{ij}_\alpha = \widetilde{D}_{(r)}(z^{ij}_\alpha) = \frac{d}{dt}(z^{ij}_\alpha \circ \tilde{\tau}^{(r)}_t)|_{t=0} =$$

$$= D^\alpha u_{ij} - \sum_{\ell} \sum_{\beta<\alpha} \binom{\alpha}{\beta} z^{ij}_{\beta+(\ell)} D^{\alpha-\beta} u_\ell$$

and from the formula for u_{ij} in (1.1.4) we obtain:

$$(2.2.2) \qquad u^{ij}_\alpha = \sum_\ell \left(\sum_{\beta\leqslant\alpha} \binom{\alpha}{\beta} z^{\ell j}_\beta D^{\alpha-\beta+(\ell)}(u_i) - \right.$$

$$\left. - \sum_{\beta<\alpha} \binom{\alpha}{\beta} z^{ij}_{\beta+(\ell)} D^{\alpha-\beta}(u_\ell) \right)$$

The right hand side in (2.2.2) at a point $j^r_x s$ only depends on $D^\sigma u_j(x)$, $|\sigma| \leqslant r+1$; that is, $u^{ij}_\alpha(j^r_x s)$ only depends on $j^{r+1}_x(D)$, thus proving (2.2.1). We shall now prove the last three statements of the theorem.

(i) We set $s_{ij} = z_{ij} \circ s$. We proceed by recurrence on r. For $r = 0$, the condition $\widetilde{D}(s(x)) = 0$ is equivalent to $u_j(x) = 0$ and $u_{ij}(x) = \Sigma_h s_{hj}(x) \, (\partial u_i/\partial x_h)\,(x) = 0$, by virtue of (1.1.4). Since the matrix $(s_{hj}(x))$ is invertible, from the second equation we obtain $(\partial u_i/\partial x_h)\,(x) = 0$; hence $j^1_x(D) = 0$. Let us assume that the relation $\widetilde{D}_{(r-1)}\,(j^{r-1}_x s) = 0$ implies $j^r_x(D) = 0$ for some $r > 0$. If $\widetilde{D}_{(r)}\,(j^r_x s) = 0$, according to (1.3.3) - (b) we also have $\widetilde{D}_{(r-1)}\,(j^{r-1}_x s) = 0$; thus by the recurrence hypothesis it follows that $D^\sigma u_i(x) = 0$ for all $|\sigma| \leqslant r$, and from (2.2.2) for $|\alpha| = r$ we obtain:

$$u^{ij}_\alpha \,(j^r_x s) = \sum_{\ell} s_{\ell j}(x)\,(D^{\alpha + (\ell)}\, u_i)\,(x) = 0.$$

Since $(s_{\ell j}(x))$ is invertible, $(D^{\alpha+(\ell)}u_i)(x) = 0$ for $|\alpha| = r$; i.e., $(D^\sigma u_i)\,(x) = 0$ for $|\sigma| = r + 1$. Therefore, $j^{r+1}_x(D) = 0$.

(ii) Since $\mathscr{M}^r$ is the sheaf of (differentiable) sections of a vector bundle, it is clear that $\mathscr{M}^r$ is locally free, and since ϕ_r is injective we have: rank of $\mathscr{M}^r$ = dimension of the fibres of $J^{r+1}(TX) = n \begin{pmatrix} n+r+1 \\ r+1 \end{pmatrix}$. Moreover, the sections $j^{r+1}(D)$ span $J^{r+1}(TX)$; thus their images, $\widetilde{D}_{(r)}$, span $\mathscr{M}^r$. Therefore, if D, D' are two vector fields of $\mathscr{M}^r$, we can write $D = \Sigma\, f_i \widetilde{D}^i_{(r)}$, $D' = \Sigma\, f_j' \widetilde{D}'^j_{(r)}$ and then from (1.3.3) - (c) we obtain

$$[D, D'] = \sum_j (\widetilde{D} f_j')\, D'^j_{(r)}$$

$$- \sum_i (D' f_i)\, \widetilde{D}^i_{(r)} + \sum_{i,j} f_i f_j' \,[D^i, D'^j]_{(r)}$$

thus showing that $\mathscr{M}^r$ is involutive.

(iii) This follows immediately from definition (2.1) and the fact that $\mathscr{M}^r$ is spanned by the vector fields $\widetilde{D}_{(r)}$. ∎

3. A CANONICAL PROCEDURE TO OBTAIN NEW DIFFERENTIAL INVARIANTS

(3.1). With the notations as in (1.2), let us denote by $J^\infty(Y)$ the projective limit of the system $(J^r(Y), \pi^r_k)$ endowed with the sheaf of rings $\mathscr{A}$ = direct limit of $(\pi^\infty_k)^* C^\infty_{J^k(Y)}$, where $\pi^\infty_k : J^\infty(Y) \to J^k(Y)$ stands for the canonical projection. An algebraic derivation (over $\mathbb{R}$) of the sheaf of rings $\mathscr{A}$ is said to be a vector field on $J^\infty(Y)$. We can also define differential forms, etc. on $J^\infty(Y)$.

We denote by $\hat{D}$ the formal lift of a vector field D on the base manifold X to $J^\infty(Y)$. In a fibred coordinate system (x_j, y_i) we thus have:

$$\hat{\partial}/\partial x_j = \partial/\partial x_j + \sum_i \sum_{|\alpha|=0}^{\infty} y^i_{\alpha+(j)} (\partial/\partial y^i_\alpha). \tag{3.1.1}$$

Let D be a vector field on Y. Since $D_{(r)}$ projects onto $D_{(r-1)}$, the system $(D_{(r)})$ defines a vector field $D_{(\infty)}$ on $J^\infty(Y)$ that is completely determined by the condition: $D_{(\infty)}(f \circ \pi^\infty_r) = (D_{(r)}f) \circ \pi^\infty_r$ for all $f \in C^\infty(J^r(Y))$.

If $f : Y \to Y'$ is a fibred mapping for which $\bar{f}$ is a diffeomorphism (notations as in (1.2)), then the induced mappings $f^{(r)} : J^r(Y) \to J^r(Y')$ define a morphism of ringed spaces $f^{(\infty)} : (J^\infty(Y), \mathscr{A}) \to (J^\infty(Y'), \mathscr{A}')$. These considerations will be mainly applied to $Y = L(X)$.

THEOREM (3.2). *There exist globally defined vector fields* $\mathbb{D}_1, \ldots, \mathbb{D}_n$ *on* $J^\infty(L(X))$ *uniquely determined by the following conditions:*

(a) $\theta^r(\mathbb{D}_j) = 0$ *for all positive integer* r *and* $j = 1, \ldots, n$, *where* θ^r *is the structure form on* $J^r(L(X))$.

(b) $\theta_i(\mathbb{D}_j) = \delta_{ij}$, *where* $\theta_1, \ldots, \theta_n$ *are the components of the canonical form on* $L(X)$.

Furthermore, such vector fields fulfil the following properties:

(i) $\mathbb{D}_j$ *maps* $C^\infty(J^r(L(X)))$ *into* $C^\infty(J^{r+1}(L(X)))$. *In fact, if* $f \in C^\infty(J^r(L(X)))$ *and* $s(x) = (D^1_x, \ldots, D^n_x)$ *we have:*

$$(\mathbb{D}_j f)(j^{r+1}_x s) = D^j_x(f \circ j^r s), \quad 1 \leqslant j \leqslant n. \tag{3.2.1}$$

(ii) The vector fields $\mathbb{D}_j$ *commute with the lifting* $\widetilde{D}_{(\infty)}$; *i.e.,* $[\widetilde{D}_{(\infty)}, \mathbb{D}_j] = 0$ *for every vector field* D *on* X. *Hence, if* f *is an* r*-order differential invariant, the functions* $\mathbb{D}_1 f, \ldots, \mathbb{D}_n f$ *are differential invariants of order* $r+1$.

Proof. The structure form of $J^r(L(X))$ is given by $\theta^r = \Sigma_{i,j} \Sigma_{|\alpha|<r} \theta^{ij}_\alpha \otimes \partial/\partial z^{ij}_\alpha$ where $\theta^{ij}_\alpha = dz^{ij}_\alpha - \Sigma_k z^{ij}_{\alpha+(k)} dx_k$, as follows from (1.3.1) and (1.1). By imposing condition (a) and (b) of the statement on an arbitrary vector field $\mathbb{D}_j = \Sigma_i \lambda_{ij} \partial/\partial x_i + \Sigma_{h,i} \Sigma_\alpha \lambda^{hi}_{\alpha j} \partial/\partial z^{hi}_\alpha$ we find,

$$\theta^{hi}_\alpha(\mathbb{D}_j) = \lambda^{hi}_{\alpha j} - \Sigma_k \lambda_{kj} z^{hi}_{\alpha+(k)} = 0,$$

$$\theta_i(\mathbb{D}_j) = \Sigma_k \lambda_{kj} z^{ik} = \delta_{ij}.$$

From the second equation we obtain $\lambda_{ij} = z_{ij}$, and by substituting into the first equation we have $\lambda^{hi}_{\alpha j} = \Sigma_k z_{kj} z^{hi}_{\alpha+(k)}$. Hence,

$$\mathbb{D}_j = \Sigma_i z_{ij} (\partial/\partial x_i + \Sigma_{h,k} \Sigma_\alpha z^{hk}_{\alpha+(i)} (\partial/\partial z^{hk}_\alpha)) \tag{3.2.2}$$

$= \Sigma_i z_{ij} (\hat{\partial}/\partial x_i)$.

With the hypothesis of (i) we now have:

$$(\mathbb{D}_j f)(j_x^{r+1} s) = \Sigma_i z_{ij}(s(x))(\hat{\partial} f/\partial x_i)(j_x^{r+1} s)$$
$$= \Sigma_i z_{ij}(s(x))(\partial/\partial x_i)(f \circ j^r s)(x) = D_x^j (f \circ j_x^r s).$$

Furthermore, from (1.3.2) it follows that $L_{\widetilde{D}(\infty)} \theta^r = A \circ \theta^r$. Thus,

$$(L_{\widetilde{D}_{(\infty)}} \theta^r)(\mathbb{D}_j) = \nabla_{\widetilde{D}_{(\infty)}} \theta^r(\mathbb{D}_j) - \theta^r([\widetilde{D}_{(\infty)}, \mathbb{D}_j]) = A(\theta^r(\mathbb{D}_j)),$$

and hence $\theta^r([\widetilde{D}_{(\infty)}, \mathbb{D}_j]) = 0$; or equivalently, $\theta_\alpha^{ij}([\widetilde{D}_{(\infty)}, \mathbb{D}_j]) = 0$. Moreover, from (1.1.3)-(b) we have $L_{\widetilde{D}(\infty)} \theta_i = 0$ and thus, $\widetilde{D}_{(\infty)} \theta_i(\mathbb{D}_j) - \theta_i([\widetilde{D}_{(\infty)}, \mathbb{D}_j]) = 0$; that is, $\theta_i([\widetilde{D}_{(\infty)}, \mathbb{D}_j]) = 0$. Since $(\theta_\alpha^{ij}, \theta_i)$ is a local basis of the differentials of $J^\infty(L(X))$, we conclude that $[\widetilde{D}_{(\infty)}, \mathbb{D}_j] = 0$. The last part of (ii) is now immediate, thus finishing the proof of the theorem. ∎

REMARK (3.3). Using (3.2.1), it is also easy to see that $\widetilde{\tau}^{(\infty)} \cdot \mathbb{D}_j = \mathbb{D}_j$ for any diffeomorphism τ of X. Actually, if f is a differentiable function on $J^r(L(X))$ and $s = (D^1, \ldots, D^n)$, then we have:

$$(\widetilde{\tau}^{(\infty)} \cdot \mathbb{D}_j)(f)(j_x^{r+1} s) = \mathbb{D}_j (f \circ \widetilde{\tau}^{(r)})(j_{\tau^{-1}(x)}^{r+1}(\widetilde{\tau}^{-1} \circ s \circ \tau))$$
$$= (\tau^{-1} \cdot D^j)_{\tau^{-1}(x)}(f \circ j^r s \circ \tau) = D_x^j(f \circ j^r s) = (\mathbb{D}_j f)(j_x^{r+1} s).$$

4. THE BASIS OF THE RING OF DIFFERENTIAL INVARIANTS

(4.1). As is well-known, there exists an affine bundle $K(X) \to X$ modeled over the vector bundle $T^*(X) \otimes T^*(X) \otimes T(X)$ whose global (differentiable) sections can be identified with the linear connections of X; we can thus consider the r-jet extension $j_x^r(\nabla)$ of a linear connection ∇ at a point $x \in X$.

The proof of the following proposition is straightforward.

PROPOSITION (4.2). *Let ∇ be a linear connection of X, and T and R the torsion and curvature of ∇, respectively. We have:*

(i) $j_x^{r-1}(\nabla_D D')$ only depends on $j_x^{r-1}(D)$, $j_x^r(D')$ and $j_x^{r-1}(\nabla)$.

(ii) $j_x^r(T)$ only depends on $j_x^r(\nabla)$.

(iii) $j_x^{r-1}(R)$ only depends on $j_x^r(\nabla)$.

(iv) $(\nabla^r T_p^q)_x$ only depends on $j_x^r(T_p^q)$ and $j_x^{r-1}(\nabla)$, where T_p^q is a tensor field on X of covariant degree p and contravariant degree q. ∎

(4.3). Every basis $D^1, \ldots, D^n$ of the module of vector fields on an open subset U of X defines a linear connection ∇ on U by the condition

$$(4.3.1) \qquad \nabla_{D^i} D^j = 0, \quad 1 \leqslant i, \ j \leqslant n.$$

Let $s : U \to L(X)$ be the section determined by the above frame; i.e., $s(x) = (D^1_x, \ldots, D^n_x), x \in U$. If U is a coordinate domain with coordinates $(x_1, \ldots, x_n)$, and we set $s_{ij} = z_{ij} \circ s$, then it is easy to see that the components of ∇ are given by,

$$(4.3.2) \qquad \Gamma^i_{jk} = -\sum_h (\partial s_{ih}/\partial x_j)\, s^{hk}.$$

These formulas show that $j^{r-1}_x(\nabla)$ only depends on $j^r_x s$. We call $j^{r-1}_x(\nabla)$ the $(r-1)$-jet of connection associated with the r-jet of frame $j^r_x s$.

According to (4.3.1), the curvature of ∇ vanishes and the torsion is given by

$$(4.3.3) \qquad T = -\sum_{j\ k} \omega^j \wedge \omega^k \otimes [D^j, D^k],$$

where $\omega^1, \ldots, \omega^n$ is the dual coframe.

DEFINITION (4.4). *Let us define functions* $f^i_{jk} : J^1(L(X)) \to \mathbb{R}$ $(i, j, k = 1, \ldots, n)$ *as follows:* $f^i_{jk}(j^1_x s)$ *are the components of the torsion* T *at* $x \in X$ *of the linear connection* ∇ *defined by the frame* s *in the basis* $s(x) = (D^1_x, \ldots, D^n_x)$; *that is,*

$$(4.4.1) \qquad T(D^j_x, D^k_x) = \sum_i f^i_{jk}(x)\, D^i_x.$$

Recall that $T_x = j^0_x(T)$ only depends on $\nabla_x = j^0_x(\nabla)$, and $j^0_x(\nabla)$ only depends on $j^1_x s$ ((4.2) - (ii) and (4.3.2)).

PROPOSITION (4.5). *The functions* f^i_{jk} *is the previous definition are first order differential invariants satisfying* :

$$(4.5.1) \qquad [\mathbb{D}_j, \mathbb{D}_k] + \sum_i f^i_{jk}\, \mathbb{D}_i = 0,$$

where $\mathbb{D}_1, \ldots, \mathbb{D}_n$ *are the vector fields introduced in (3.2).*

Proof. As usual, we set $s_{ij} = z_{ij}\, s$. From (4.3.3) and (4.4.1) we obtain

$$f^i_{jk}(j^1_x s) = \sum_{h,\ell} (s_{hk}(x)\, (\partial s_{\ell j}/\partial x_h)(x) -$$

$$- s_{hj}(x)\,(\partial s_{\ell k}/\partial x_h)(x))\, s^{i\ell}(x).$$

That is,

$$f^i_{jk} = \sum_{h,\ell} (z_{hk}\, z^{\ell j}_{(h)} \quad z_{hj}\, z^{\ell k}_{(h)}) z^{i\ell}, \tag{4.5.2}$$

which proves that f^i_{jk} is a differentiable function on $J^1\ (L(X))$. Moreover, since $[\hat{\partial}/\partial x_j, \hat{\partial}/\partial x_k] = 0$, from (3.2.2) and (3.1.1) we deduce:

$$[\mathbb{D}_j, \mathbb{D}_k] = \sum_{h,\ell} (z_{hj}\, z^{\ell k}_{(h)} - z_{hk}\, z^{\ell j}_{(h)}\ \hat{\partial}/\partial x_\ell .$$

Hence,

$$[\mathbb{D}_j, \mathbb{D}_k] = -\sum_i \sum_{h,\ell} (z_{hk}\, z^{\ell j}_{(h)} - z_{hj}\, z^{\ell k}_{(h)})\, z^{i\ell}\, \mathbb{D}_i = -\sum_i f^i_{jk} \mathbb{D}_i,$$

by virtue of (4.5.2).

Finally, for any vector field D on X, (4.5.1) yields

$$[\widetilde{D}_{(\infty)}, [\mathbb{D}_j, \mathbb{D}_k]] + \sum_i (\widetilde{D}_{(1)} f^i_{jk})\, \mathbb{D}_i + \sum_i f^i_{jk} [\widetilde{D}_{(\infty)}, \mathbb{D}_i] = 0,$$

and the first and the last terms vanish because of Jacobi identity and (3.2)-(ii). Therefore, $\widetilde{D}_{(1)} f^i_{jk} = 0$. ■

DEFINITION (4.6). *For each $r \in \mathbb{N}$ let us define functions*

$$f^i_{j_1 \ldots j_r k \ell} : J^{r+1}(L(X)) \to \mathbb{R} \quad (i, j_1, \ldots, j_r, k, \ell = 1, \ldots, n)$$

as follows: $f^i_{j_1 \ldots j_r k \ell}(j^{r+1}_x s)$ are the components of the tensor $(\nabla^r T)_x$ in the basis $s(x) = (D^1_x, \ldots, D^n_x)$, where T is the torsion of the linear connection ∇ defined by the frame s; that is,

$$(\nabla^r T)(D^{j_1}_x, \ldots, D^{j_r}_x, D^k_x, D^\ell_x) = \sum_i f^i_{j_1 \ldots j_r k \ell}(j^{r+1}_x s)\, D^i_x . \tag{4.6.1}$$

REMARK (4.6.2). The definition is correct, because $(\nabla^r T)_x$ only depends on $j^r_x(\nabla)$ and $j^r_x(\nabla)$ only depends on $j^{r+1}_x s$ ((4.2)-(ii) and (4.3.2)). Note also that for $r = 0$ the above definition coincides with (4.4). Accordingly, we agree that $f^i_{j_1 \ldots j_r k \ell}$ for $r = 0$ means $f^i_{k\ell}$.

PROPOSITION (4.7).

(i) $f^i_{j_1 \dots j_r k \ell} = \mathbb{D}_{j_1} \dots \mathbb{D}_{j_r} (f^i_{k \ell})$.

(ii) $f^i_{j_1 \dots j_r k \ell}$ *is a differential invariant of order* $r + 1$.

Proof. Proceeding by recurrence on r, it will be sufficient to prove that $f^i_{j_1 \dots j_r k \ell} = \mathbb{D}_{j_1} (f^i_{j_2 \dots j_r k \ell})$. Let us assume $s = (D^1, \dots, D^n)$ on an open subset U. By virtue of (4.3.1) we have:

$$\sum_i f^i_{j_1 \dots j_r k \ell} (j^{r+1}_x s) D^i_x =$$

$$= \nabla_{D^{j_1}_x} ((\nabla^{r-1} T)(D^{j_2}, \dots, D^{j_r}, D^k, D^\ell)).$$

From the very definition of the functions $f^i_{j_2 \dots j_r k \ell}$, it follows that on U we have:

$$(\nabla^{r-1} T)(D^{j_2}, \dots, D^{j_r}, D^k, D^\ell) = \sum_i (f^i_{j_2 \dots j_r k \ell} \circ j^r s) D^i,$$

and again from (4.3.1),

$$\nabla_{D^{j_1}_x} ((\nabla^{r-1} T)(D^{j_2}, \dots, D^{j_r}, D^k, D^\ell)) =$$

$$= \sum_i D^{j_1}_x (f^i_{j_2 \dots j_r k \ell} \circ j^r s) D^i_x$$

$$= \sum_i f^i_{j_1 \dots j_r k \ell} (j^{r+1}_x s) D^i_x .$$

Hence,

$$f^i_{j_1 \dots j_r k \ell} (j^{r+1}_x s) = D^{j_1}_x (f^i_{j_2 \dots j_r k \ell} \circ j^r s) =$$
$$= (\mathbb{D}_{j_1} f^i_{j_2 \dots j_r k \ell}) (j^{r+1}_x s)$$

by virtue of (3.2.1). Part (ii) now follows from part (i), Prop. (4.5) and (3.2)-(ii). ■

THEOREM (4.8). *For every positive integer* r, *let* F_r *be the set of functions* $f^i_{j_1 \dots j_m k \ell}$ *defined in (4.6) whose indices satisfy the following conditions:*

(a) $0 \leqslant m \leqslant r - 1$,

(b) $j_1 \geqslant \dots \geqslant j_m \geqslant k$, and $k < \ell$.

Then:

(i) The number of functions of F_r is $N_{n,r} = \dim J^r(L(X)) - rk\,\mathcal{M}^r$.

(ii) The differentials of the functions of F_r are linearly independent at every point of $J^r(L(X))$.

(iii) Every differential invariant of order r can be locally written as a differentiable function of the functions of F_r.

Proof. (i) Let I_m be the set of the system of indices $(j_1, \ldots, j_m, k, \ell)$ such that: $j_1 \geqslant \ldots \geqslant j_m \geqslant k$, $j_1, \ldots, j_m, k; \ell = 1, \ldots, n$, and I'_m the subset of I_m defined by: $j_1 \geqslant \ldots \geqslant j_m \geqslant k \geqslant \ell$. For each m, the system of indices in $I_m \quad I'_m$ are exactly those which fulfil conditions (a) and (b) of the statement. The number of elements of I_m is $n\binom{n+m}{n+1}$, and that of I'_m is $\binom{n+m+1}{m+2}$. Hence, the number of functions of F_r is

$$\sum_{m=0}^{r-1} \left(n^2 \binom{n+m}{m+1} - n \binom{n+m+1}{m+2} \right) =$$

$$= n^2 \binom{n+r}{r} + n - n \binom{n+r+1}{r+1} = \dim J^r(L(X)) - rk\,\mathcal{M}^r,$$

as follows from (2.2) - (ii) and the identity

$$\sum_{k=0}^{r} \binom{n+k-1}{k} = \binom{n+r}{r}.$$

(ii) Let $(U; x_1, \ldots, x_n)$ be a coordinate open subset of X. Set $J^r = J^r(L(U))$. Starting with (3.2.2) and by recurrence on r, it is easily checked that there exist functions $F^{\alpha}_{j_1 \ldots j_{r-1}} \in C^\infty(J^{r-1})$ such that,

$$(*) \qquad \mathbb{D}_{j_1} \circ \ldots \circ \mathbb{D}_{j_{r-1}} =$$

$$= \sum_{i_1, \ldots, i_{r-1}} (z_{i_1 j_1} \ldots z_{i_{r-1} j_{r-1}})\, \hat{\partial}/\partial x_{i_1} \circ \ldots \circ \hat{\partial}/\partial x_{i_{r-1}}$$

$$+ \sum_{|\alpha| < r-1} F^{\alpha}_{j_1 \ldots j_{r-1}} (\hat{\partial}/\partial x)^{\alpha},$$

where $(\hat{\partial}/\partial x)^\alpha = (\hat{\partial}/\partial x_1)^{\alpha_1} \ldots (\hat{\partial}/\partial x_n)^{\alpha_n}$. One should recall that $[\hat{\partial}/\partial x_j, \hat{\partial}/\partial x_k] = 0$, in such a way that the compositions of the vector fields $\hat{\partial}/\partial x_j$ can be dealt with like polynomials.

Similarly, starting with (4.5.2) and by recurrence on r, it is also seen that there exist functions $G^i_{j_1 \ldots j_{r-1} k \ell} \in C^\infty(J^{r-1})$ such that,

$$(**)\qquad (\hat{\partial}/\partial x_{i_1} \circ \ldots \circ \hat{\partial}/\partial x_{i_{r-1}})(f^i_{k\ell}) =$$

$$\sum_{h,m} (z_{h\ell}\, z^{mk}_{(j_1 \ldots j_{r-1} h)} - z_{hk}\, z^{m\ell}_{(j_1 \ldots j_{r-1} h)})\, z^{im} + G^i_{j_1 \ldots j_{r-1} k\ell}.$$

Let us denote by $\overline{df}$ the restriction of df to the vector sub-bundle $T(J^r/J^{r-1})$, for every $f \in C^\infty(J^r)$. Hence, $\overline{d(gf)} = g\,\overline{df}$ whenever $g \in C^\infty(J^{r-1})$. From (*) and (**) we now obtain

$$(+)\qquad \overline{df^i_{j_1 \ldots j_{r-1} k\ell}} =$$

$$\sum_{i_1, \ldots, i_{r-1}} \sum_{h,m} (z_{i_1 j_1} \ldots z_{i_{r-1} j_{r-1}}\, z_{h\ell}\, z^{im})\, \overline{dz^{mk}_{(j_1 \ldots j_{r-1} h)}} -$$

$$- \sum_{i_1, \ldots, i_{r-1}} \sum_{h,m} (z_{i_1 j_1} \ldots z_{i_{r-1} j_{r-1}}\, z_{hk}\, z^{im})\, \overline{dz^{m\ell}_{(j_1 \ldots j_{r-1} h)}},$$

for all $r \geqslant 2$.

Given a point $j^r_x s \in J^r$, let us choose a coordinate open neighbourhood $(U; x_1, \ldots, x_n)$ of x, contained within the domain of s, such that $s(x) = (D^1_x, \ldots, D^n_x) = ((\partial/\partial x_1)_x, \ldots, (\partial/\partial x_n)_x)$; or equivalently, $z_{ij}(s(x)) = \delta_{ij}$. We shall now prove that the differentials of the functions of F_r are linearly independent at $j^r_x s$ by recurrence on r. Assume $\Sigma_i \Sigma_{k<\ell} \lambda^i_{k\ell}\, d_{j^1_x s} f^i_{k\ell} = 0$ for $r = 1$. Restricting to $T_{j^1_x s}(J^1/J^0)$, from (4.5.2) and the properties of the coordinate system chosen, we obtain $\Sigma_i \Sigma_{k<\ell} \lambda^i_{k\ell} (d_{j^1_x s} z^{ik}_{(\ell)} - d_{j^1_x s} z^{i\ell}_{(k)}) = 0$. Hence, $\lambda^i_{k\ell} = 0$. Now, let us assume

$$\sum_{m=0}^{r-1} \sum_i \sum_{k<\ell} \sum_{j_1 \geqslant \ldots \geqslant j_m \geqslant k} \lambda^i_{j_1 \ldots j_m k\ell} (d_{j^r_x s} f^i_{j_1 \ldots j_m k\ell}) = 0$$

for $r \geqslant 2$.

By restricting to $T_{j^r_x s}(J^r/J^{r-1})$, we obtain:

$$(++)\qquad \sum_i \sum_{k<\ell} \sum_{j_1 \geqslant \ldots \geqslant j_{r-1} \geqslant k} \lambda^i_{j_1 \ldots j_{r-1} k\ell}\, \overline{d_{j^r_x s} f^i_{j_1 \ldots j_{r-1} k\ell}} = 0,$$

because $f^i_{j_1 \ldots j_m k\ell}$ belongs to $C^\infty(J^{r-1})$ for $m < r-1$. Thus, by virtue of the recurrence hypothesis it will be sufficient to prove that $\lambda^i_{j_1 \ldots j_{r-1} k\ell} = 0$. From (+) and the properties of the coordinate system chosen, we obtain

$$(\mathrm{x})\qquad \overline{d_{j^r_x s} f^i_{j_1 \ldots j_{r-1} k\ell}} = \overline{d_{j^r_x s} z^{ik}_{(j_1 \ldots j_{r-1} \ell)}} - \overline{d_{j^r_x s} z^{i\ell}_{(j_1 \ldots j_{r-1} k)}}.$$

Let us consider two systems of indices h, b, c, $a_1, \ldots, a_{r-1}$ and i, k, ℓ, $j_1, \ldots, j_{r-1}$ such that $b < c$, $a_1 \geqslant \ldots \geqslant a_{r-1} \geqslant b$ and $k < \ell$, $j_1 \geqslant \ldots \geqslant j_{r-1} \geqslant k$. We have:

(1) $\partial z^{ik}_{(j_1 \ldots j_{r-1}\ell)} / \partial z^{hc}_{(a_1 \ldots a_{r-1}b)} = 0$, otherwise it would be $h = i$, $c = k$, $(a_1 \ldots a_{r-1}\, b) = (j_1 \ldots j_{r-1}\, \ell)$. However, the inequalities $j_1 \geqslant \ldots \geqslant j_{r-1} \geqslant k = c > b$, $\ell > k = c > b$ imply that the first b components of the multi-index $(j_1 \ldots j_{r-1}\ell)$ vanish, thus contradicting that $(j_1 \ldots j_{r-1}) = (a_1 \ldots a_{r-1} b)$.

(2) $\partial z^{i\ell}_{(j_1 \ldots j_{r-1}k)} / \partial z^{hc}_{(a_1 \ldots a_{r-1}b)} \neq 0$ if and only if $h = i$, $c = \ell$, $a_1 = j_1$, $\ldots, a_{r-1} = j_{r-1}$, $b = k$, since $(a_1 \ldots a_{r-1}\, b) = (j_1 \ldots j_{r-1}\, k)$ and $a_1 \geqslant \ldots \geqslant a_{r-1} \geqslant b$, $j_1 \geqslant \ldots \geqslant j_{r-1} \geqslant k$ imply $a_1 = j_1, \ldots, a_{r-1} = j_{r-1}$, $b = k$.

Formula (x) together with (1) and (2) show that by evaluating (++) at the tangent vector $(\partial/\partial z^{hc}_{(a_1 \ldots a_{r-1}b)})_{j^r_x s}$ we obtain $\lambda^h_{a_1 \ldots a_{r-1}bc} = 0$.

Part (iii) follows from parts (i) and (ii), and from (4.7)-(ii). ∎

5. THE GLOBAL STRUCTURE OF THE RINGS OF INVARIANTS

DEFINITION (5.1). *A differentiable function $f : J^r(L(X)) \to \mathbb{R}$ is said to be* invariant under diffeomorphisms *if $f \circ \tilde{\tau}^{(r)} = f$ for every diffeomorphism τ of X.*

The set of r-order invariants under diffeomorphisms constitutes a subring A'_r of $C^\infty(J^r(L(X)))$.

PROPOSITION (5.2). *Every invariant under diffeomorphisms is a differential invariant; in other words, $A'_r \subset A_r$.*

Proof. Let $f : J^r(L(X)) \to \mathbb{R}$ be an invariant under diffeomorphisms. Every vector field D on X with compact support generates a global 1-parameter group of diffeomorphisms τ_t. Since f is invariant by $\tilde{\tau}^{(r)}_t$, we have $\widetilde{D}_{(r)} f = 0$, and since the vector fields with compact support are dense in the set of vector fields of X with respect to the C^∞ topology, we conclude that $\widetilde{D}_{(r)} f = 0$ for every vector field D on X. ∎

PROPOSITION (5.3). *The functions $f^i_{j_1 \ldots j_r k \ell}$ introduced in (4.6) are invariant under diffeomorphisms.*

Proof. According to (4.7)-(i), it will be sufficient to prove the following two properties:

(5.3.1) If f is invariant under diffeomorphisms, then $\mathbb{D}_j f$ is also invariant under diffeomorphisms.

(5.3.2) $f^i_{k\ell}$ is invariant under diffeomorphisms.

The second statement follows by transforming (4.5.1) by $\tilde{\tau}^{(\infty)}$, taking into account that $\tilde{\tau}^{(\infty)} \cdot \mathbb{D}_j = \mathbb{D}_j$ for every diffeomorphism τ of X. As for (5.3.1), according to (3.2.1) we have:

$$((\mathbb{D}_j f) \circ \tilde{\tau}^{(r+1)})\ (j^{r+1}_x s) = (\tau \cdot D^j)_{\tau(x)}(f \circ j^r(\tilde{\tau} \circ s \circ \tau^{-1}))$$
$$= (\tau \cdot D^j)_{\tau(x)}(f \circ \tilde{\tau}^{(r)} \circ j^r s \circ \tau^{-1}) = (\tau \cdot D^j)_{\tau(x)}(f \circ j^r s \circ \tau^{-1})$$
$$= D^j_x (f \circ j^r s) = (\mathbb{D}_j f)\ (j^{r+1}_x s). \qquad \blacksquare$$

(5.4). Let G^r_x be the set of the r-jet extensions $j^r_x \tau$ at $x \in X$ of invertible (differentiable) map-germs $\tau : (X, x) \to (X, x)$. It is not difficult to prove that G^r_x is a Lie group with respect to the topology induced from $J^r(X, X)$ and the composition of jets, of dimension $n \binom{n+r}{r} - n$, which has two connected components. Its identity component, consisting of the elements of positive determinant, will be denoted by $\mathring{G}^r_x$. We also have a canonical projection of Lie groups $\pi^r_k : G^r_x \to G^k_x$ for $r \geqslant k$.

Since $\tilde{\tau}^{(r)}\ (j^r_x s) = j^r_x(\tilde{\tau} \circ s \circ \tau^{-1})$ for every invertible map-germ $\tau : (X, x) \to$ $\to (X, x)$, it is clear that $\tilde{\tau}^{(r)}\ (j^r_x s)$ only depends on $j^{r+1}_x \tau$. We can thus define a (differentiable) action on the right of G^{r+1}_x on $J^r_x(L(X))$, the fibre of $p_{(r)}$ over $x \in X$, by setting $j^r_x s \cdot j^{r+1}_x \tau = (\tilde{\tau}^{-1})^{(r)}\ (j^r_x s)$.

Let

$$N_{n,r} = \dim J^r(L(X)) - rk\mathscr{M}^r$$
$$= n^2 \binom{n+r}{r} + n - n \binom{n+r+1}{r+1}$$

be the number of functions of F_r (see (4.8) and (2.2)-(ii)). We denote by $t^i_{j_1 \dots j_m k\ell}$, with $0 \leqslant m \leqslant r-1$, $j_1 \geqslant \dots \geqslant j_m \geqslant k$, and $k < \ell$ (conditions (a) and (b) in (4.8)), the coordinates of $\mathbb{R}^{N_{n,r}}$ and by

(5.4.1) $$\pi^r : J^r(L(X)) \to \mathbb{R}^{N_{n,r}}$$

the mapping whose components are the functions $f^i_{j_1 \dots j_m k\ell}$ of F_r; i.e.,

$$t^i_{j_1 \dots j_m k\ell} \circ \pi^r = f^i_{j_1 \dots j_m k\ell}.$$

We also write:

(5.4.2) $$\pi^r_x = \pi^r|_{J^r_x(L(X))}.$$

With the above assumptions and notations, we have the following important result which solves the problem of formal equivalence for complete parallelisms in terms of the invariants F_r, $r \in \mathbb{N}$ (the «restricted equivalence problem» in the terminology of E. Cartan), furthermore giving the structure of the corresponding moduli space:

THEOREM (5.5). *For every $x \in X$, the mapping*

$$\pi^r_x : J^r_x(L(X)) \to \mathbb{R}^{N_{n,r}}$$

defined in (5.4.2) is a principal bundle with group G^{r+1}_x.

Proof (1) *π^r_x is a submersion.* Part (ii) in (4.8) is equivalent to saying that the mapping π^r in (5.4.1) is a submersion. Moreover, from (4.5.2) and (4.7)-(i) it is easily checked that in any induced coordinate system (x_j, z^{hi}_α), $|\alpha| \leqslant r$, one has $\partial f^i_{j_1 \dots j_m k\ell}/\partial x_j = 0$ for $m < r$, thus proving that π^r_x is also a submersion.

(2) *π^r_x is surjective.* The origin of $\mathbb{R}^{N_{n,r}}$ belongs to the image of π^r_x, because if $x_1, \dots, x_n$ is a coordinate system on an open neighbourhood U of x, the torsion of the linear connection associated with the frame $(\partial/\partial x_1, \dots, \partial/\partial x_n)$ vanishes; hence,

$$f^i_{j_1 \dots j_m k\ell}(j^r_x(\partial/\partial x_1, \dots, \partial/\partial x_n)) = 0$$

for all $f^i_{j_1 \dots j_m k\ell} \in F_r$.

Let $s = (D^1, \dots, D^n)$ be an arbitrary frame on U and let $x_1, \dots, x_n$ be a coordinate system with origin x mapping U onto $\mathbb{R}^n$. If $D^j_z = \Sigma_i s_{ij}(z)(\partial/\partial x_i)_z$, $z \in U$, for every $\lambda \in \mathbb{R}$ we define a frame $s_\lambda = (D^1_\lambda, \dots, D^n_\lambda)$ by setting $(D^j_\lambda)_z = \Sigma_i s_{ij}(\lambda z)(\partial/\partial x_i)_z$. Then, $D^j_\lambda = \lambda(\tilde{\tau}^{-1}_\lambda \cdot D^j)$ for all $\lambda \neq 0$, where τ_λ is the diffeomorphism $\tau_\lambda(z) = \lambda z$. Hence $j^r_x s = (\tilde{\tau}^{-1}_\lambda)^{(r)}(j^r_x(\lambda s))$. Note that $\lambda x = x$, because x is the origin of the coordinate system. Since $f^i_{j_1 \dots j_m k\ell}$ is invariant under diffeomorphisms,

$$f^i_{j_1 \dots j_m k\ell}(j^r_x s_\lambda) = f^i_{j_1 \dots j_m k\ell}(j^r_x(\lambda s)),$$

and taking into account that the frames $s = (D^1, \dots, D^n)$ and $\lambda s = (\lambda D^1, \dots, \lambda D^n)$ have the same associated connection, from the definition of the functions $f^i_{j_1 \dots j_m k\ell}$ and the above formula we obtain:

$$f^i_{j_1 \dots j_m k\ell}(j^r_x s_\lambda) = \lambda^{m+1} f^i_{j_1 \dots j_m k\ell}(j^r_x s).$$

For any point $t = (t^i_{j_1 \dots j_m k\ell}) \in \mathbb{R}^{N_{n,r}}$ we set

$$t_\lambda = \left(\frac{1}{\lambda^{m+1}} t^i_{j_1 \dots j_m k\ell} \right) \text{ for } \lambda \in \mathbb{R}, \quad \lambda \neq 0.$$

Since $\lim_{\lambda \to \infty} t_\lambda$ is the origin of $\mathbb{R}^{N_{n,r}}$, which belongs to the image of π^r_x, and since

π^r_x is a submersion, for all large enough λ the point t_λ also belongs to the image of π^r_x. Thus, there exists $j^r_x s$ such that

$$\frac{1}{\lambda^{m+1}} t^i_{j_1 \dots j_m k \ell} = f^i_{j_1 \dots j_m k \ell}(j^r_x s).$$

Hence,

$$t^i_{j_1 \dots j_m k \ell} = \lambda^{m+1} f^i_{j_1 \dots j_m k \ell}(j^r_x s) = f^i_{j_1 \dots j_m k \ell}(j^r_x s_\lambda).$$

Consequently, $t \in Im\ \pi^r_x$.

(3) *G^{r+1}_x acts freely on $J^r_x(L(X))$.* Given a frame $s = (D^1, \dots, D^n)$, let us consider a coordinate system with origin x such that $D^j_x = (\partial/\partial x_j)_x$, $1 \leqslant j \leqslant n$. For any invertible map-germ τ at x, we have:
$\tau \cdot D^j = \Sigma_{h,i}(s_{hj} \circ \tau^{-1})((\partial\tau_i/\partial x_h) \circ \tau^{-1})\partial/\partial x_i$, where $s_{ij} = z_{ij} \circ s$, $\tau_i = x_i \circ \tau$. Thus, $\tilde{\tau}^{(r)}(j^r_x s) = j^r_x s$ if and only if:

$$\Sigma_h D^\alpha\{(s_{hj} \circ \tau^{-1})((\partial\tau_i/\partial x_h) \circ \tau^{-1})\}(x) = D^\alpha s_{ij}(x), \tag{5.5.1}$$

for $|\alpha| \leqslant r$, $1 \leqslant i, j \leqslant n$.

By recurrence on r we shall prove that (5.5.1) implies the following:

$$(F_r) \quad \tau_i = x_i + f_{ir}, \text{ with } f_{ir} \in m^{r+2}_x,$$

where m_x is the ideal of the functions of $C^\infty(X)$ vanishing at x, thus proving that $j^{r+1}_x(\tau)$ is the identity. For $r = 0$, (5.5.1) yields $(\partial\tau_i/\partial x_j)(x) = \delta_{ij}$, thus proving (F_0). Let us fix a multi-index α of order $|\alpha| = r > 0$. From (5.5.1) and Leibniz formula we obtain:

$$\sum_h \sum_{0<\beta<\alpha} \binom{\alpha}{\beta} D^{\alpha-\beta}(s_{hj} \circ \tau^{-1}) D^\beta((\partial\tau_i/\partial x_h) \circ \tau^{-1})(x) +$$
$$+ D^\alpha(s_{ij} \circ \tau^{-1})(x) + D^\alpha\{(\partial\tau_i/\partial x_j) \circ \tau^{-1}\}(x) =$$
$$= (D^\alpha s_{ij})(x), \quad |\alpha| \leqslant r, \quad 1 \leqslant i, j \leqslant n.$$

The first term on the right hand side vanishes because from the recurrence hypothesis (F_{r-1}) we conclude that

$$(\partial\tau_i/\partial x_h) \circ \tau^{-1} - \delta_{hi} = (\partial f_{i,r-1}/\partial x_h) \circ \tau^{-1}$$

belongs to m^r_x. Hence

$$D^\alpha(s_{ij} \circ \tau^{-1})(x) + D^\alpha\{(\partial\tau_i/\partial x_j) \circ \tau^{-1}\}(x) = (D^\alpha s_{ij})(x). \tag{5.5.2}$$

Moreover, the Taylor expansion of s_{ij} at x to order r can be written as follows:

$$s_{ij} \equiv \sum_{|\alpha| \leqslant r} \frac{1}{\alpha!} D^\alpha s_{ij}(x) x_1^{\alpha_1} \ldots x_n^{\alpha_n} \pmod{m_x^{r+1}}.$$

Trasforming this congruence by τ^{-1}, we have:

$$s_{ij} \circ \tau^{-1} \equiv \sum_{|\alpha| \leqslant r} \frac{1}{\alpha!} D^\alpha s_{ij}(x) x_1^{\alpha_1} \ldots x_n^{\alpha_n} \pmod{m_x^{r+1}},$$

since $x_i \circ \tau^{-1} = x_i - f_{i,r-1} \circ \tau^{-1}$ and $f_{i,r-1} \circ \tau^{-1} \in m_x^{r+1}$. Hence, $D^\alpha(s_{ij} \circ \tau^{-1})(x) = D^\alpha s_{ij}(x)$, and formula (5.5.2) allows us to finish the proof.

(4) *If* $\pi_x^r(j_x^r s) = \pi_x^r(j_x^r \bar{s})$, *there exists* $j_x^{r+1}\tau \in G_x^{r+1}$ *such that* $j_x^r s = j_x^r \bar{s} \cdot j_x^{r+1}\tau$. Let ∇, $\bar{\nabla}$ be the linear connections associated with the frames $s = (D^1, \ldots, D^n)$, $\bar{s} = (\bar{D}^1, \ldots, \bar{D}^n)$ and $(x_1, \ldots, x_n)$, $(\bar{x}_1, \ldots, \bar{x}_n)$ the normal coordinate systems determined by the frames $(D_x^1, \ldots, D_x^n)$, $(\bar{D}_x^1, \ldots, \bar{D}_x^n)$, respectively. We set: $s_{ij} = z_{ij} \circ s$, $\bar{s}_{ij} = \bar{z}_{ij} \circ \bar{s}$. With the obvious notations, we first prove:

(5.5.3) Let τ be the unique diffeomorphism such that $x_i = \bar{x}_i \circ \tau$, $1 \leqslant i \leqslant n$. Then, $\tilde{\tau}^{(r)}(j_x^r s) = j_x^r \bar{s}$ if and only if $D^\alpha s_{ij}(x) = \bar{D}^\alpha \bar{s}_{ij}(x)$ for all $|\alpha| \leqslant r$, $1 \leqslant i, j \leqslant n$.

We set $s' = (\tau \cdot D^1, \ldots, \tau \cdot D^n)$, $s'_{ij} = \bar{z}_{ij} \circ s'$. Since $\tau(x) = x$, we have $j_x^r s' = \tilde{\tau}^{(r)}(j_x^r s)$. It will thus be sufficient to show that $\bar{D}^\alpha \bar{s}'_{ij}(x) = D^\alpha s_{ij}(x)$ for $|\alpha| \leqslant r$. Actually, we shall prove that $(\bar{D}^\alpha s'_{ij}) \circ \tau = D^\alpha s_{ij}$.

Transforming by τ the identity:

$$[\partial/\partial x_{i_1}, [\ldots, [\partial/\partial x_{i_k}, D^j] \ldots] =$$

$$= \sum_i (\partial^k s_{ij}/\partial x_{i_1} \ldots \partial x_{i_k})(\partial/\partial x_i)$$

and noting that $\tau \cdot \partial/\partial x_i = \partial/\partial \bar{x}_i$, we obtain $\partial^k s'_{ij}/\partial \bar{x}_{i_1} \ldots \partial \bar{x}_{i_k} = (\partial^k s_{ij}/\partial x_{i_1} \ldots \partial x_{i_k}) \circ \tau^{-1}$ which is equivalent to our previous statement.

We now continue the proof of (4). The hypothesis of (4) means $f^i_{j_1 \ldots j_m k \ell}(j_x^r s) = f^i_{j_1 \ldots j_m k \ell}(j_x^r \bar{s})$ for all $f^i_{j_1 \ldots j_m k \ell} \in F_r$. According to (5.5.3), we shall show that this hypothesis implies: $\bar{D}^\alpha \bar{s}_{ij}(x) = D^\alpha s_{ij}(x)$ for all $|\alpha| \leqslant r$, $1 \leqslant i, j \leqslant n$. We proceed by recurrence on r. For $r = 1$ the hypothesis is $\Gamma^i_{k\ell}(x) - \Gamma^i_{\ell k}(x) = \bar{\Gamma}^i_{k\ell}(x) - \bar{\Gamma}^i_{\ell k}(x)$. As the coordinate systems are normal, we have $\Gamma^i_{k\ell}(x) + \Gamma^i_{\ell k}(x) = \bar{\Gamma}^i_{k\ell}(x) + \bar{\Gamma}^i_{\ell k}(x) = 0$. Hence $\Gamma^i_{k\ell}(x) = \bar{\Gamma}^i_{k\ell}(x)$, and the result follows from (4.3.2). Note that $s_{ij}(x) = \bar{s}_{ij}(x) = \delta_{ij}$. We can thus assume $r > 1$. Let τ be the diffeomorphism defined in (5.5.3). Transforming by $\tilde{\tau}^{(\infty)}$ the formulas (*) and (**) in the proof of (4.8) and taking into account that $\tilde{\tau}^{(\infty)} \cdot \mathbb{D}_j = \mathbb{D}_j$, $\tilde{\tau}^{(\infty)} \cdot \hat{\partial}/\partial x_j = \hat{\partial}/\partial \bar{x}_j$ (and hence $\tilde{\tau}^{(\infty)} \cdot (\hat{\partial}/\partial x)^\alpha = (\hat{\partial}/\partial \bar{x})^\alpha$),

$\bar{z}_{ij} \circ \tilde{\tau} = z_{ij}$ and the fact that $f^i_{k\ell}$ is invariant under diffeomorphisms, we obtain:

$$\bar{F}^{\alpha}_{j_1 \dots j_{r-1}} \circ \tilde{\tau}^{(r-1)} = F^{\alpha}_{j_1 \dots j_{r-1}},$$
$$\bar{G}^{i}_{j_1 \dots j_{r-1} k \ell} \circ \tilde{\tau}^{(r-1)} = G^{i}_{j_1 \dots j_{r-1} k \ell}.$$

From the hypothesis and the recurrence hypothesis we thus conclude

$$(\partial^r s_{ik}/\partial x_{j_1} \dots \partial x_{j_{r-1}} \partial x_{\ell})(x) - (\partial^r s_{i\ell}/\partial x_{j_1} \dots \partial x_{j_{r-1}} \partial x_k)(x) =$$
$$(\partial^r \bar{s}_{ik}/\partial \bar{x}_{j_1} \dots \partial \bar{x}_{j_{r-1}} \partial \bar{x})(x) - (\partial^r \bar{s}_{i\ell}/\partial \bar{x}_{j_1} \dots \partial \bar{x}_{j_{r-1}} \partial \bar{x}_k)(x),$$

or equivalently,

(I) $$(D^{\alpha-(k)} s_{ik})(x) - (D^{\alpha-(\ell)} s_{i\ell})(x) =$$
$$(D^{\alpha-(k)} s_{ik})(x) - (D^{\alpha-(\ell)} s_{i\ell})(x)$$

for all $|\alpha| = r+1$, $1 \leqslant k, \ell \leqslant n$.

Moreover, as the coordinate systems are normal, we have: $\Sigma_{j,k} \Gamma^i_{jk}(t\lambda)\lambda_j\lambda_k = \Sigma_{j,k} \bar{\Gamma}^i_{jk}(t\lambda)\lambda_j\lambda_k = 0$ for $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, $|t| < \epsilon$. From the identity

$$(d^r/dt^r)\,\Gamma^i_{jk}(t\lambda) = \sum_{|\alpha| \leqslant r} \frac{1}{\alpha!} D^{\alpha}\Gamma^i_{jk}(t\lambda)\lambda^{\alpha}$$

we obtain:

$$\sum_{j,k} \frac{1}{(\alpha-(jk))!} (D^{\alpha-(jk)}\Gamma^i_{jk})(x) = 0$$

for all $|\alpha| \geqslant 2$, and similarly for the $\bar{\Gamma}^i_{jk}$'s. Using (4.3.2), we finally obtain:

$$\Sigma_k \alpha_k (D^{\alpha-(k)} s_{ik})(x) =$$
$$\sum_{h,j,k} \sum_{0 < \beta \leqslant \alpha-(jk)} (\alpha!/(\alpha-\beta-(jk))!\,\beta!(1-|\alpha|))$$
$$(D^{\alpha-\beta-(k)} s_{ih})(x)\,(D^{\beta} s^{hk})(x),$$

where we have used $\Sigma_i\, 1/(\alpha-(i))! = |\alpha|/\alpha!$, and similarly for s. If $|\alpha| = r+1$, by virtue of the recurrence hypothesis from the above equation we deduce:

(II) $$\sum_k \alpha_k (D^{\alpha-(k)} s_{ik})(x) = \sum_k \alpha_k (D^{\alpha-(k)} s_{ik})(x).$$

It is now easy to see that (I) and (II) imply $(D^{\alpha-(\ell)} s_{i\ell})(x) = (\bar{D}^{\alpha-(\ell)}\bar{s}_{i\ell})(x)$

for all $\ell = 1, \ldots, n$, thus finishing the proof of the theorem. ■

COROLLARY (5.6) (Local equivalence for complete parallelisms in the analytic case). *Let $s = (D^1, \ldots, D^n)$, $\bar{s} = (\bar{D}^1, \ldots, \bar{D}^n)$ be two frames of an analytic manifold X defined on some neighbourhood of a point $x \in X$. There then exists an invertible (analytic) map-germ $\tau : (X, x) \to (X, x)$ such that $\tau \cdot D^j = D^j$, $1 \leqslant j \leqslant n$, if and only if $f^i_{j_1 \ldots j_m k \ell}(j^r_x s) = f^i_{j_1 \ldots j_m k \ell}(j^r_x s)$ for all $f^i_{j_1 \ldots j_m k \ell} \in \in F_r$, $r \in \mathbb{N}$.*

Proof. Let us use the notations of part (4) in the proof of th. (5.5). The diffeomorphism τ in (5.5.3) is now an analytic isomorphism such that $\tilde{\tau}^{(r)}(j^r_x s) = = j^r_x(\tau \cdot D^1, \ldots, \tau \cdot D^n) = j^r_x(\bar{D}^1, \ldots : \bar{D}^n) = j^r_x \bar{s}$ for all $r \in \mathbb{N}$. Since $\tau \cdot D^j$, D^j are analytic, they are determined by $j^\infty_x (\tau \cdot D^j)$, $j^\infty_x (\bar{D}^j)$, respectively. ■

Remark. Given a frame $s = (D^1, \ldots, D^n)$ defined on an open neighbourhood U_x of a point $x \in X$, the restrictions of the invariants $f^i_{j_1 \ldots j_m k \ell}$, $m \leqslant r$ (defined in (4.6)) to $j^{r+1}_x s$ are precisely the families of functions F_r associated with s in [10], pag. 342. If the families F_r and $\bar{F}_r$ corresponding to the frames $s = (D^1, \ldots, D^n)$ and $\bar{s} = (\bar{D}^1, \ldots, \bar{D}^n)$ (both defined on some neighbourhood of x) satisfy the conditions of theorem 4.1 in [10], pag. 344, then it is clear that our conditions for formal equivalence ($f^i_{j_1 \ldots j_m k \ell}(j^r_x s) = f^i_{j_1 \ldots j_m k \ell}(j^r_x \bar{s})$, for all $f^i_{j_1 \ldots j_m k \ell} \in F_r$) are also satisfied. Since in the analytic case, formal equivalence implies local equivalence, the converse is also true in this case; but regrettably this is not the case for C^∞ equivalence.

We recall that a connected manifold is called *reversible* if it is orientable and admits an orientation reversing diffeomorphism.

THEOREM (5.7) (Structure of the rings A'_r, A_r) *Let π^r be the projection defined in (5.4.1). We have:*

(1) If X is non-orientable, then $A'_r = A_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}})$.

(2) If X is orientable and non-reversible, then

$$A'_r = A_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}) \oplus (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}).$$

(3) If X is reversible, then

$$A'_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}),$$

$$A_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}) \oplus (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}).$$

the inclusion $A'_r \subset A_r$ being the diagonal mapping.

Proof. With the notations as in (5.4), we set $K^r_x = Ker\ \pi^r_1$; that is, K^r_x is the group of the r-jets $j^r_x \tau$ of invertible map-germs τ whose Jacobian matrix at x is the identity. It is easily verified that K^r_x is a connected Lie group.

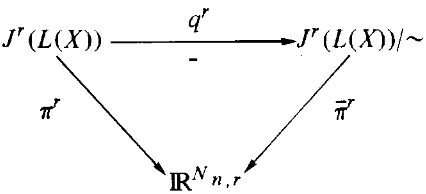
Let us fix a point $t \in \mathbb{R}^{N_{n,r}}$, and let $\eta^r : (\pi^r)^{-1}(t) \to L(X)$ be the mapping $\eta^r(j^r_x s) = s(x)$. By using the above theorem, it is not difficult to see that η^r is a fibre bundle whose fibre over a point $s(x) \in L(X)$ is diffeomorphic to K^{r+1}_x. Hence, $(\pi^r)^{-1}(t)$ is connected if and only if X is non-orientable, and if X is orientable, then $(\pi^r)^{-1}(t)$ has two connected components.

Moreover, according to (4.8), the vector fields of the differential system $\mathcal{M}^r$ are the vector fields on $J^r(L(X))$ tangent to the fibres of the submersion π^r. Therefore, A_r is the ring of differentiable functions on $J^r(L(X))$ which are constant on the connected components of the fibres of π^r.

If X is non-orientable, it thus follows that A_r consists precisely of the functions in $C^\infty(J^r(L(X)))$ which are constant on each of the fibres of π^r; or equivalently, $A_r = (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}})$. This formula shows that A_r is differentiably generated by the components $f^i_{j_1 \dots j_m k \ell}$ of π^r, which are functions invariant under diffeomorphisms. Hence, $A'_r = A_r$.

Let us now assume that the manifold X is orientable, and let $\sim$ be the equivalence relation on $J^r(L(X))$ defined by: $j^r_x s \sim j^r_{x'} s'$ if and only if $\pi^r(j^r_x s) = \pi^r(j^r_{x'} s') = t$, and $j^r_x s$ and $j^r_{x'} s'$ belong to the same connected component of $(\pi^r)^{-1}(t)$.

The projection $\pi^r : J^r(L(X)) \to \mathbb{R}^{N_{n,r}}$ induces a 2-sheet covering $\bar{\pi}^r : J^r(L(X))/\sim \to \mathbb{R}^{N_{n,r}}$ making the diagram commutative



where the horizontal arrow stands for the canonical projection. Since $\bar{\pi}^r$ is trivial, we have:

$$A_r = (q^r)^* C^\infty(J^r(L(X))/\sim) = (q^r)^* C^\infty(C^+) \oplus (q^r)^* C^\infty(C^-)$$
$$= (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}) \oplus (\pi^r)^* C^\infty(\mathbb{R}^{N_{n,r}}),$$

C^+, C^- being the connected components of $J^r(L(X))/\sim$. If $\tilde{\tau}^{(r)}((q^r)^{-1}C^+) = (q^r)^{-1}C^+$ and $\tilde{\tau}^{(r)}((q^r)^{-1}C^-) = (q^r)^{-1}C^-$ for every diffeomorphism τ of X (i.e., if X is non-reversible), then every function in A_r is invariant under diffeomorphisms; hence $A'_r = A_r$. If X is orientable, there exists a diffeomorphism τ such that $\tilde{\tau}^{(r)}((q^r)^{-1}C^+) = (q^r)^{-1}C^-$, and a function $f \in A_r$ is invariant under diffeomorphisms if and only if $f \circ \tilde{\tau}^{(r)} = f$. Part (3) of the theorem now follows by identifying $(q^r)^{-1}C^-$ with $(q^r)^{-1}C^+$ via $\tilde{\tau}^{(r)}$, thus finishing the proof. ∎

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